

Macro - 7020: TA Session 2

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Taylor Expansion

- ▶ Taylor's theorem tells us the following

$$f(x_t) = f(x) + f'(x)(x_t - x) + \frac{f^{(2)}(x)}{2!}(x_t - x)^2 + \frac{f^{(3)}(x)}{3!}(x_t - x)^3 + \dots$$

where the expansion is considered "at x "

- ▶ usually we expand around the steady state (x^*) in the context of Macro
- ▶ For smooth functions, the magnitude of the terms dissipates quickly with n
- ▶ So the bumbling idiots in economics usually feel they can simply write

$$f(x_t) = f(x) + f'(x)(x_t - x) \text{ and } f(x_t, y_t) = f(x, y) + f_x(x, y)(x_t - x) + f_y(x, y)(y_t - y)$$

where equality is imposed but it's really an approximation.

Log(-)Linearization

- ▶ The usual definition for a (log) linearized variable is $\widehat{x}_t = \frac{x_t - x}{x}$
 - ▶ Think about this as relative deviation from the steady state.
 - ▶ Value: non-relative numbers are arbitrary. Also, cycles and shocks.
- ▶ First order Taylor expansion about $\widehat{x}_t = 0$

$$\ln(1 + \widehat{x}_t) \approx \ln(1) + \left(\frac{d}{d\widehat{x}_t} \ln(1 + \widehat{x}_t) \Big|_{\widehat{x}_t=0} \right) (\widehat{x}_t - 0) = \left(\frac{1}{1 + \widehat{x}_t} \Big|_{\widehat{x}_t=0} \right) \widehat{x}_t = \widehat{x}_t$$

"valid" since we consider \widehat{x}_t to be small in magnitude

- ▶ So now consider the following property, which is extremely useful

$$\widehat{x}_t \approx \ln(1 + \widehat{x}_t) = \ln \left(1 + \frac{x_t - x}{x} \right) = \ln \left(\frac{x_t}{x} \right) = \ln(x_t) - \ln(x)$$

Still Log-Linearizing

- ▶ Now we have $\widehat{x}_t \approx \ln(x_t) - \ln(x)$. So for instance

$$y_t = x_t z_t \implies \widehat{y}_t \approx \ln(y_t) - \ln(y) = (\ln(x_t) + \ln(z_t)) - (\ln(x) + \ln(z)) \approx \widehat{x}_t + \widehat{z}_t$$

This gives us the first in several rules

- ▶ $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$ *product rule*
 - ▶ $y_t = x_t^\alpha \implies \widehat{y}_t = \alpha \widehat{x}_t$ *power rule*
 - ▶ $y_t = f(x_t) \implies \widehat{y}_t = \left[\frac{f'(x)}{f(x)} x \right] \widehat{x}_t$ *function rule*
 - ▶ $y_t = x_t + z_t \implies \widehat{y}_t = x \widehat{x}_t + \widehat{z}_t$ *sum rule*
- ▶ These are: incredibly useful, all you need, and (mostly) transparent
 - ▶ Implicit, (mathematically) trivial, but important rule: linearized constant = 0
 - ▶ Rule 3 Proof [▶ Appendix](#)

The Brutal Truth about LL..

- ▶ Taylor series expansions in econ are **not** about mathematical precision
 - ▶ Jensen's inequality? Never heard of her
- ▶ Write equality signs. Just do it.
 - ▶ The last slide I went from \approx to $=$ when writing "rules". I'm never going back
 - ▶ For this year, don't worry. But for future: "does this matter?" is good for research
- ▶ To add onto the ingrained grainyness..
 - ▶ "black box": people say log-linearize and magically a solution appears
 - ▶ Because there's lots of messy math behind the scenes, can be hard to implement
- ▶ I have been taught a ton of different ways to do this over the years. Here are in my opinion the best two: rule based and brute force

Method 1: Rule Based (or The Method of Big Hat™)

- ▶ Intuitive and step by step: just apply the rules over and over

- ▶ $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$

- ▶ $y_t = x_t^\alpha \implies \widehat{y}_t = \alpha \widehat{x}_t$

- ▶ $y_t = f(x_t) \implies \widehat{y}_t = \left[\frac{f'(x)}{f(x)} x \right] \widehat{x}_t$

- ▶ $y_t = x_t + z_t \implies y \widehat{y}_t = x \widehat{x}_t + z \widehat{z}_t$

- ▶ Example: Consider $k_{t+1} = (1 - \delta)k_t + sA_t k_t^\alpha$. This means

$$\widehat{k}_{t+1} = \frac{(1 - \delta)k}{k} (\widehat{(1 - \delta)k_t}) + \frac{sAk^\alpha}{k} \widehat{sA_t k_t^\alpha} \quad \text{Rule 4}$$

$$= (1 - \delta) [(\widehat{(1 - \delta)}) + \widehat{k}_t] + sAk^{\alpha-1} (\widehat{s} + \widehat{A}_t + \widehat{k}_t^\alpha) \quad \text{Rule 1}$$

$$= (1 - \delta) \widehat{k}_t + sAk^{\alpha-1} (\widehat{A}_t + \alpha \widehat{k}_t) \quad \text{Rule 2}$$

- ▶ key is treating entire term as one linearized variable and then "shrinking the hat"

Method 2: Brute Force

- ▶ Just compute (some intuition from Taylor def but gets fuzzy when thinking about constants)

$$0 = f(x_t, y_t, z_t) \implies 0 = f_x(x, y, z)x\hat{x}_t + f_y(x, y, z)y\hat{y}_t + f_z(x, y, z)z\hat{z}_t$$

- ▶ Breaking this down: set everything equal to 0. Call this expression $f(\dots)$
 - ▶ Say you have k variables $\{x_{i,t}\}_{i=1}^k$
 - ▶ Log-linearizing is

$$0 = \sum_{i=1}^k f_{x_i}(SS) \cdot x_i \hat{x}_{i,t}$$

- ▶ Each term: partial derivative of f w.r.t x_i evaluated at the steady states (x_1, \dots, x_k) multiplied by $x_i \cdot \hat{x}_{i,t}$ (steady state of x_i times linearized x_i)
- ▶ Returning to our $k_{t+1} = (1 - \delta)k_t + sA_t k_t^\alpha$ example (so $0 = -k_{t+1} + (1 - \delta)k_t + sA_t k_t^\alpha$)

$$\begin{aligned} 0 &= -1 \times k\hat{k}_{t+1} + (1 - \delta + \alpha s A k^{\alpha-1}) \times k\hat{k}_t + s k^\alpha \times A\hat{A}_t \\ &\implies \hat{k}_t = (1 - \delta)\hat{k}_t + s A k^{\alpha-1}(\hat{A}_t + \alpha\hat{k}_t) \end{aligned}$$

Which method do I use?

- ▶ Try on your own and see what your brain likes best
- ▶ For most people, brute force will be best (cleaner)
- ▶ But always remember the rules! Brute force can be cumbersome in simple cases
- ▶ Say you want to linearize δx_t
 - ▶ Rules-based immediately gives you \hat{x}_t (just use product rule)
 - ▶ Remember: "hat" treats all objects equally. Can't think about constants until it's isolated
 - ▶ Brute force only really makes sense with an equation. So you have to redefine $y_t = \delta x_t$

$$\implies 0 = -y\hat{y}_t + \delta x\hat{x}_t$$

and then you have to realize/recognize the steady state of y_t is δx

- ▶ "Realizing" is often an essential simplifying step and source of struggle with LL

Practice!

► For reference, recall: $0 = \sum_{i=1}^k f_{x_i}(SS) \cdot x_i \widehat{x}_{i,t}$

► $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$

► $y_t = x_t^\alpha \implies \widehat{y}_t = \alpha \widehat{x}_t$

► $y_t = f(x_t) \implies \widehat{y}_t = \left[\frac{f'(x)}{f(x)} x \right] \widehat{x}_t$

► $y_t = x_t + z_t \implies y \widehat{y}_t = x \widehat{x}_t + z \widehat{z}_t$

1. $y_t = -x_t$

2. $y_t = (x_t + \beta z_t)^\alpha$

3. $c_t + k_{t+1} - (1 - \delta)k_t = A_t k_t^\alpha \ell_t^{1-\alpha}$

4. How does #3 simplify if you know steady states $c = \gamma k$ and $A, \ell = 1$

5. $c_{t+1} = \beta [c_t (\alpha A_t k_{t+1}^{\alpha-1} + 1 - \delta)]$

Proof of Rule 3 [▶ Back to Rules](#)

- ▶ A simple way to see this is

$$\begin{aligned}\hat{y}_t &\approx \ln(y_t) - \ln(y) = \ln(f(x_t)) - \ln(f(x)) \approx \left[\ln(f(x)) + \frac{f'(x)}{f(x)}(x_t - x) \right] - \ln(f(x)) \\ &= \frac{f'(x)}{f(x)}(x_t - x) = \frac{xf'(x)}{f(x)}\hat{x}_t\end{aligned}$$

- ▶ We can also consider $f(x_t) = \frac{g(x_t)}{h(x_t)} \implies \ln(f(x_t)) = \ln(g(x_t)) - \ln(h(x_t))$. So

$$\ln(f(x_t)) = \ln(f(x)) + \frac{f'(x)}{f(x)}(x_t - x) \quad \ln(g(x_t)) = \ln(g(x)) + \frac{g'(x)}{g(x)}(x_t - x)$$

$$\ln(h(x_t)) = \ln(h(x)) + \frac{h'(x)}{h(x)}(x_t - x)$$

- ▶ Combing these Taylor expansions with $\ln(f(x_t)) = \ln(g(x_t)) - \ln(h(x_t))$ yields

$$\frac{f'(x)}{f(x)}(x_t - x) = \frac{g'(x)}{g(x)}(x_t - x) - \frac{h'(x)}{h(x)}(x_t - x) \implies \frac{xf'(x)}{f(x)}\hat{x}_t = \frac{xg'(x)}{g(x)}\hat{x}_t - \frac{xh'(x)}{h(x)}\hat{x}_t$$